

## Optimal response function in networks of excitatory elements

Zbysław Tabor\*

*Department of Biophysics, Jagiellonian University Medical College, ul. Grzegorzewska 16a, 31-531 Cracow, Poland*  
(Received 30 March 2006; revised manuscript received 31 May 2006; published 2 August 2006)

In this paper the problem of signal propagation in networks of excitatory elements is studied. It is found that the geometry of signal transmission paths depends crucially on how an excitatory element responds to a stimulus. Two types of responses are defined: fast and slow. In the slow response case the signal transmission paths are in the same universality class as optimal paths in the limit of strong disorder. The signal transmission paths formed in the fast response case constitute possibly a new universality class.

DOI: [10.1103/PhysRevE.74.021102](https://doi.org/10.1103/PhysRevE.74.021102)

PACS number(s): 05.40.-a, 64.60.Ak, 89.75.Hc

### INTRODUCTION

Transport in disordered media is an important field of statistical physics, related to many applications of practical relevance. Frequently, real-world problems involve an excitable network: a disordered lattice of bonds with randomly distributed thresholds  $V_i$ . A bond is open for transport, provided that the potential drop across it exceeds threshold value  $V_i$  and is closed otherwise. Problems of this kind arise in networks of diodes [1], flow of Bingham plastic [2], or foam [3] in porous media and dielectric breakdown [4]. In the latter case an element which has been opened for transport cannot be closed any more—an insulating element, after it has become broken, transforms irreversibly into a conductor (or superconductor). An important area of application of excitatory networks is also modeling neuronal networks, in which case additional degrees of freedom are assigned to the network elements [5].

In the present study I investigate the geometry of the signal transmission paths in a minimal model of an excitatory network, because it straightforwardly determines the time of the reaction of the network to a stimulus. The geometry of the signal transmission paths in excitatory networks is also a subject of confusion. While in some studies it is conjectured that the signal transmission paths are in the same universality class as minimal paths in percolating cluster, others claimed that it is not [3]. Here I show that the geometry of signal transmission paths depends crucially at least on the features of the response function of a single element to a stimulus.

Understanding the role of the response function for the signal propagation in heterogeneous networks of excitatory elements is important because there are neural disorders in which this function can be influenced. For example gradual destruction of myelin, related to multiple sclerosis, disturbs pulse propagation along axons [6]. Also decreased amount of neurotransmitters (e.g., reduction in dopamine, observed in Parkinson's disease [7]) can modify the reaction of a neuron to a stimulus. In this study I show on a simple model that slowed response to a stimulus can dramatically decrease functionality of a whole network, while firing response is necessary to generate an optimal reaction.

### PROBLEM FORMULATION

Here I define an excitatory element as an element, which conducts a received signal if its amplitude is larger than the threshold value and is an ideal insulator otherwise. I further assume that the resistance of an excitatory element is equal to zero in the conducting state. The values of the excitation threshold are sampled from uniform distribution. Excitatory elements are assembled and form a lattice. In this study I investigate the geometry of signal transmission paths in two-dimensional (2D) square lattices (assuming periodic boundary conditions) and in complex networks of Erdős-Rényi [8] and Watts-Strogatz type [9]. A path is a sequence of contiguous lattice sites. Given two lattice sites  $X$  and  $Y$ , the minimal signal amplitude  $E$  necessary to enable transmission from  $X$  to  $Y$  is equal to

$$E = \min_{P(X,Y)} \left[ \max_{Z \in P(X,Y)} T(Z) \right], \quad (1)$$

where  $P(X, Y)$  denotes any path, connecting  $X$  and  $Y$ ,  $Z$  is a lattice site contained in  $P$ , and  $T(Z)$  is the excitation threshold of  $Z$ .

An additional feature, which is introduced in the model of an excitatory network, is an element response function. Specifically, I consider two types of response. In a fast response case, a signal at the outputs of an excitatory element jumps from zero to  $A$  after receiving at input a signal with super-threshold amplitude  $A$ . In a slow response case a signal at the outputs of an excited element grows continuously from zero to  $A$  in a finite time.

In a case of a realistic situation the transmission of a signal between two excitatory elements is not immediate (e.g., transmission of a signal between two neurons through a synapse). Thus a fast response in a realistic network should be understood not as immediate but rather as faster than a typical time of communication between neighboring elements. Consequently, slow response should be understood as a response slower than a typical time of communication between neighboring elements.

While the shape of the response function obviously does not change the amplitude of a signal, necessary to enable transmission between some two selected sites, it can potentially change the geometry of the signal transmission paths. In the case of slow response, an excited element always transmits the received signal to this neighboring element, which has the lowest excitation threshold. In the case of fast

\*Email address: [tabor@alphas.if.uj.edu.pl](mailto:tabor@alphas.if.uj.edu.pl)

TABLE I. The values of the signal transferring exponent  $d_{STP}$ .

Lattice type	Slow response	Fast response
Square 2D	$0.610 \pm 0.002$	$0.537 \pm 0.002$
Erdős-Rényi networks	$0.355 \pm 0.003$	$0.177 \pm 0.003$
Watts-Strogatz networks	$0.353 \pm 0.003$	$0.181 \pm 0.004$

response, an excited element can transmit the signal to any of its neighbors, having excitation threshold not higher than the amplitude of the received signal.

I use the Dijkstra’s algorithm [10] to find signal transmission paths. The Dijkstra’s algorithm proceeds as follows. One starts from a seed site, which by definition has the lowest possible excitation threshold. All the neighbors of the seed are put into a queue, which I call “the invasion front,” and are labeled with their excitation thresholds. The distance of these sites from the seed is set equal to 1. Then the site  $C_{MIN}$  in the invasion front, which has the lowest label, is selected. In the case of slow response, all unlabeled neighbors of  $C_{MIN}$  are labeled with their excitation thresholds and put into the invasion front. In the case of fast response, all unlabeled neighbors of  $C_{MIN}$  are labeled with either their excitation thresholds—if it is larger than the label of  $C_{MIN}$ —or otherwise with the label of  $C_{MIN}$ . Next, these sites are put into the invasion front. The distance of these sites from the seed is equal to the distance of  $C_{MIN}$  from the seed increased by 1. Finally,  $C_{MIN}$  is removed from the invasion front. Then again a site with minimal label within the invasion front is selected and the procedure is iterated until all the lattice sites have been labeled.

After labeling all sites I calculate mean length of paths connecting lattice sites with the seed. This procedure is repeated  $10^4$  times for every lattice topology and for every lattice size.

It appears that the most expensive part of the algorithm is localization of the site with the minimal label within the invasion front. This problem can be however greatly simplified, if it is noticed that the exact values of the excitation thresholds are not in fact important. Thus, the sites of an examined lattice can be ranked with integer numbers in an order determined by the values of the excitation threshold. Then the algorithm refers only to these integer indexes, what speeds up the performance, provided an appropriate data coding is used.

**RESULTS**

The mean length  $\langle l_{STP} \rangle$  of signal transferring paths scales with the number  $N$  of the nodes of the underlying lattices:  $\langle l_{STP} \rangle \sim N^{d_{STP}}$ . Logarithm of  $\langle l_{STP} \rangle$  is plotted vs logarithm of  $N$  in Fig. 1 for square 2D, Erdős-Rényi (mean node degree  $\langle k \rangle = 12$ ) and Watts-Strogatz networks (mean node degree  $\langle k \rangle = 4$ , rerouting probability  $p = 0.05$ ). From these plots I calculate the values of the signal transferring exponent  $d_{STP}$ , listed in Table I. The values obtained for complex networks are universal and do not depend on their properties (that is on

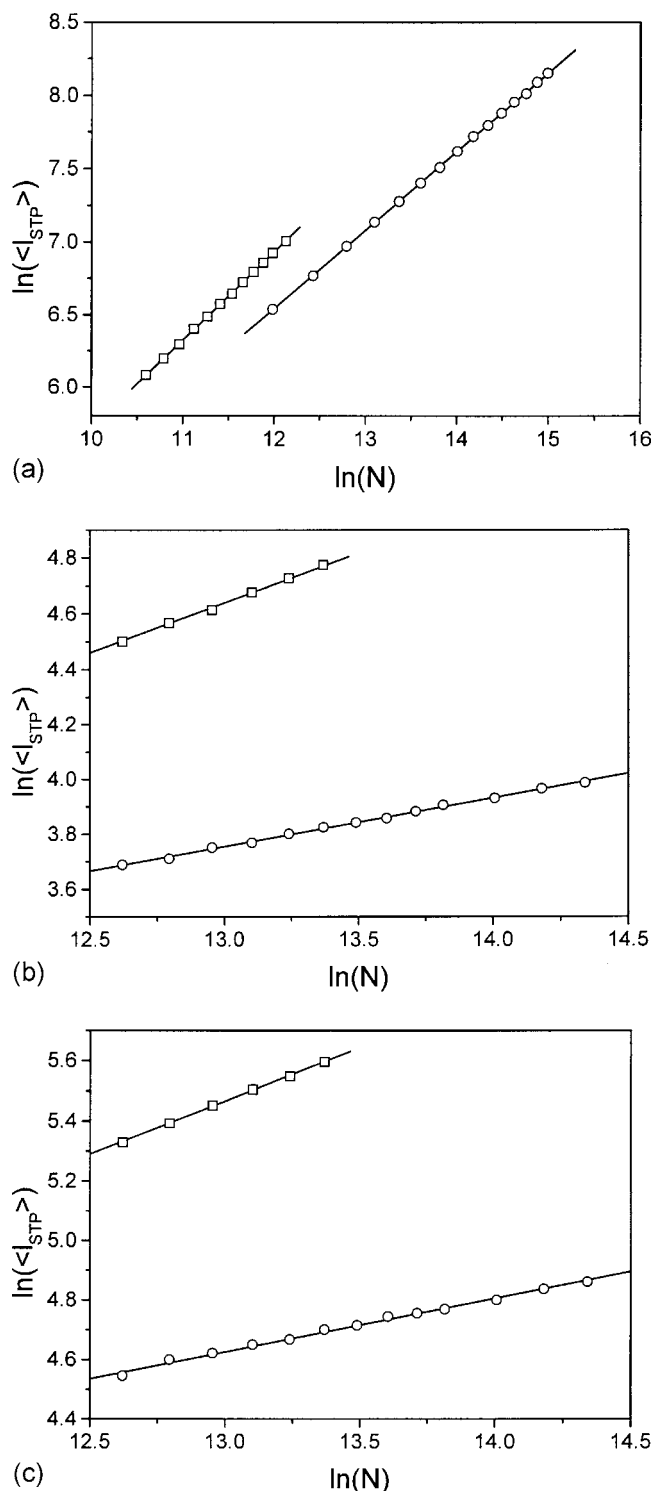


FIG. 1. Logarithm of mean length of signal transferring paths  $\langle l_{STP} \rangle$  plotted vs logarithm of the number of nodes  $N$  of underlying lattices for slow response (squares) and fast response (circles). (a) 2D square lattice. (b) Erdős-Rényi network. (c) Watts-Strogatz network.

$\langle k \rangle$  and  $p$ ). These values were obtained for the power relation fitted in the ranges of  $N$  as shown in Fig. 1.

The values of  $d_{STP}$ , found for the slow response case agree for every lattice topology with the values of the optimal path exponent reported for the strong disorder limit

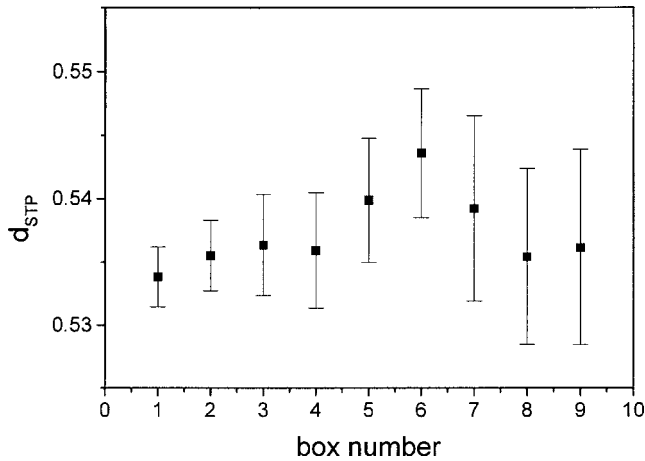


FIG. 2. Estimates of  $d_{\text{STP}}$  exponent obtained for different ranges of the number of nodes  $N$  of a 2D square lattice.

(0.61 and 0.33 for 2D square and Erdős-Rényi or Watts-Strogatz networks, respectively) [11,12]. From the description of the algorithm it follows that the model of excitation network with slow response is equivalent to the variation of the Eden growth model, developed by Cieplak *et al.* [13].

The signal transferring paths found for the fast response case are much more compact than in the case of slow response. The paths are shorter on average than in the slow response case (Fig. 1), and also their length grows more slowly with increasing lattice size (Table I).

The values of  $d_{\text{STP}}$  in the fast response case suggest possibly new universality class of the geometry of the paths. To test this hypothesis, it must be carefully checked, whether there is a possibility that the obtained results are consistent with one of the known models.

In the case of 2D square lattice the value of  $2 \times d_{\text{STP}} \approx 1.07$  in the fast response case is higher than the fractal exponent of the optimal paths in the limit of weak disorder  $d_{\text{wd}}=1$  [14,15], but smaller than the smallest fractal exponent reported for 2D site percolation models on a square lattice  $d_{\text{MIN}} \approx 1.13$  (the fractal exponent of minimal self-avoiding walks) [16,17]. To provide evidence that the observed difference is not an effect of a finite size of simulated systems, I have fitted the relation  $\langle l_{\text{STP}} \rangle \sim N^{d_{\text{STP}}}$  for every seven consecutive data points (circles) in Fig. 1(a). The results of this analysis, presented in Fig. 2 [where the box number equal to  $k$  denotes the range of data points from  $k$ th to  $(k+6)$ th], indicate that the estimate of  $d_{\text{STP}}$  is stable and thus the observed difference between  $d_{\text{STP}}$  and the fractal exponents reported for other models is probably not a finite size effect.

To further strengthen the above conclusion I have calculated the roughness exponent for the signal transferring paths on 2D square lattices in the fast response case. For this purpose I monitor the  $y$  displacement  $s$  of the end point of this path, which first reaches a distance  $L$  from the seed site along axis  $x$ . The roughness exponent is defined through the relation  $\langle s^2 \rangle \propto L^{2\nu}$ . In the model of optimal paths in the limit of weak disorder  $\nu$  is equal to  $2/3$  [14,15]. In contrast, I have found  $\nu \approx 1.0$  for signal transferring paths in the case of fast response (Fig. 3).

The above result indicates that the signal transferring paths on a 2D square lattice in the case of fast response are

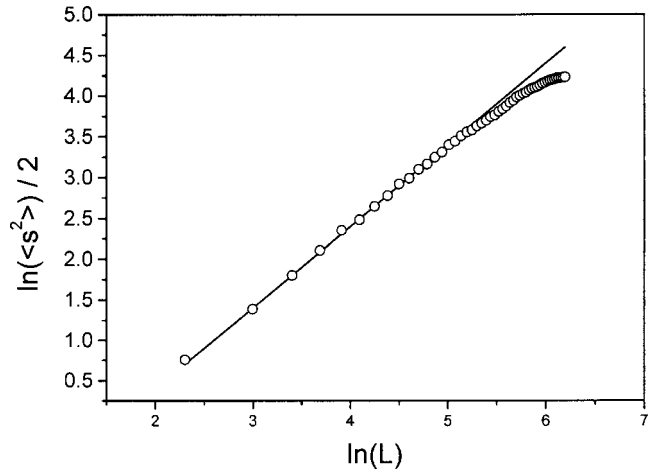


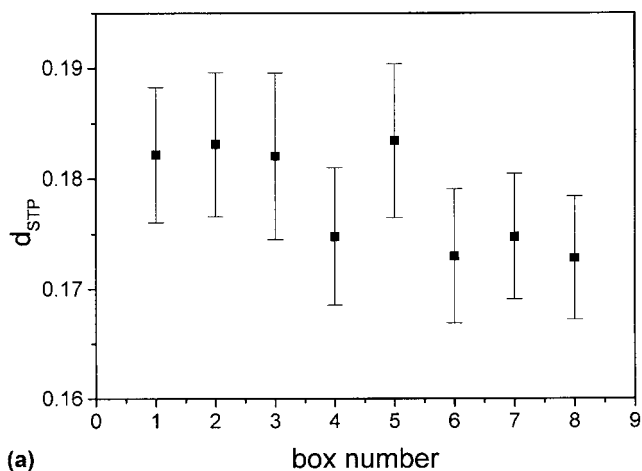
FIG. 3. Logarithm of the average squared  $y$  displacement  $\langle s^2 \rangle$  of signal transferring paths in fast response case plotted vs logarithm of  $x$  displacement  $L$  from the seed site. The data was obtained for 10 000 realization of disorder in a 2D square lattice, linear size of which was equal to 1000. Straight line with slope equal to 1 is shown for visual comparison.

different from the model of optimal paths in the limit of weak disorder. Consequently, this result must also hold for complex lattices. Numerical proof of such a statement would be however much more difficult than in the case of 2D square lattices. Proving that the length of signal transferring paths does not depend logarithmically on the number of lattice sites (such as in the case of weak disordered optimal paths) would require probing very large systems, which is unfeasible because of computational expense of generating large disordered complex lattices. In my simulations I was able to probe systems with a number of nodes up to about  $1.5 \times 10^6$ . For small systems estimates of  $d_{\text{STP}}$  depend on the range from which  $d_{\text{STP}}$  is fitted, but for larger systems  $d_{\text{STP}}$  is quite stable and stays close to  $1/6$  (Fig. 4). These results suggest that the model of excitatory network with fast response is not related to the percolation theory, because  $d_{\text{STP}}$  equal to  $1/3$  would be expected if the signal transmission paths were in the same universality class as minimal paths (or any other path model) in percolating cluster in critical dimension  $d_C=6$ . I expect that in the limit of large lattices  $1/6$  can be an exact value of  $d_{\text{STP}}$ , that is signal transferring paths are in the same universality class as the shortest paths on a hypercubic lattice in critical dimension  $d_C=6$ .

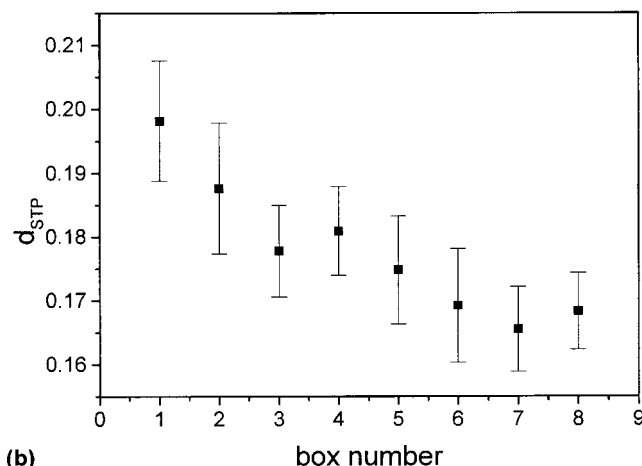
Combining the results obtained for the 2D and complex network I conjecture that the model of signal transferring paths with fast response constitutes a new universality class of path geometry.

## FINAL REMARKS

In this study I have examined a minimal model of excitatory network. According to the developed model the transition of an element from an insulating to a conducting state is determined only by an element response function. Depending on the features of the response of an excitatory element to a stimulus I have shown that there are two geometrically



(a)



(b)

FIG. 4. Estimates of  $d_{STP}$  exponent obtained for different ranges of the number of nodes  $N$  of Erdős-Rényi (a) and Watts-Strogatz (b) networks.

different classes of paths, which are equivalent in the sense of Eq. (1). The paths generated in the case of fast response are however shorter and their length grows more slowly with increasing lattice size, than in the case of slow response. I have shown that the latter model is equivalent to the model of optimal paths in the limit of strong disorder.

Now, let me discuss how the path geometry influences functionality of a network. Functionality of an excitatory network transmitting a signal can be connected to at least two factors: the input signal amplitude necessary to enable transmission through the network and the mean reaction time. The input signal amplitude is given by Eq. (1) and, in the case of the analyzed models, does not provide sufficient information to quantify network functionality. The reaction time can be estimated for example as the average time of propagation of a signal from the source node to all the nodes of the network. In the analyzed models time is not explicitly involved. However I measure the distance from the source node to all the nodes of the network. The distance from the source node can be identified with the propagation time if it is assumed that a node, which has received a signal, transmits it to all its neighbors, which have excitation thresholds smaller than the amplitude of a signal. It is true for both fast and slow re-

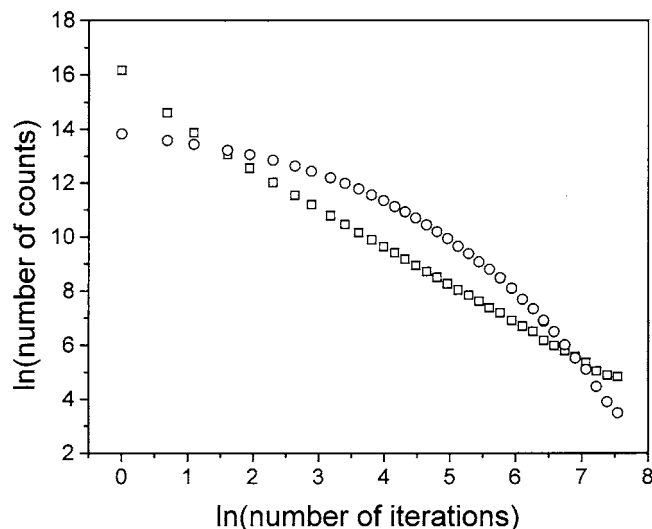


FIG. 5. Distribution of the number of iterations of the Dijkstra's algorithm for initial stage of signal propagation in 2D square lattice for slow response (squares) and fast response (circles) models.

sponse. In the latter case the nodes with lower excitation threshold transmit signals preferentially, while transmission is simultaneous in the fast response case. I disregard delays due to slow response when counting time in different nodes. Potentially, these delays, when accumulated, may lead to desynchronization of the signals propagating along different paths. The extent of desynchronization must depend on the specific form of an element response function and this process can be disregarded, provided accumulated delays are on average small comparing to a unit of time.

As long as one concerns the mean reaction time as the criterion of functionality, an excitatory network built of elements slowly responding to excitation indeed is not optimal. To enable an optimal response to an external stimulus, an excitatory element must fire, just like a neuron.

To explain the difference in the mean reaction time between the models of slow and fast response I monitored for every node the number of iterations of the Dijkstra's algorithm, which were completed since the moment of node addition to the moment of node removal from the invasion front. To avoid finite size effects I have collected the values of the number of iterations, obtained only during initial stages of signal propagation within large 2D lattices (because of technical limitations, mainly on the size of the lattices, I was not able to collect reliable—that is not disturbed by finite size effects—distribution data for initial stages of signal propagation in complex lattices). I have found that there is a qualitative difference between the distributions of the number of iterations in both models (Fig. 5). In the case of slow response the distribution of the number of iterations develops a heavy tail with exponent approximately equal to 1.4. Contrarily, the distribution of the number of iterations is exponential for the model with fast response. These results resemble the properties of priority queues, recently discussed in Ref. [18].

The shape of the distribution of the number of iterations delivers yet another criterion of the network functionality. In the case of slow response only relatively small (comparing to

the fast response) amount of nodes participates in the signal transmission, which is a disadvantage in a case when node failures can occur.

The strong disorder limit is often explained as transition from narrow to broad distribution of weights (excitation thresholds) associated with the lattice sites [19]. The model of excitatory network with fast response is a broad distribu-

tion limit of the familiar model of self-avoiding walks. If the same excitation threshold is assigned to all sites of an examined lattice (narrow distribution of excitation thresholds) then, by construction of the path generating algorithm, the signal transferring paths in an excitatory network with fast response are in the same universality class as self-avoiding walks and optimal paths in the weak disorder limit [14,15].

- 
- [1] S. Roux, A. Hansen, and E. Guyon, *J. Phys. A* **48**, 2125 (1987); S. Roux and H. J. Hermann, *Europhys. Lett.* **4**, 1227 (1987); L. Benguigui, *Phys. Rev. B* **38**, 7211 (1988).
- [2] P. M. Adler and H. Brenner, *PCH, PhysicoChem. Hydrodyn.* **5**, 287 (1984); P. G. de Gennes, *Rev. Inst. Fr. Pet.* **47**, 249 (1992).
- [3] W. R. Rossen and C. K. Mamun, *Phys. Rev. B* **47**, 11815 (1993); H. Kharabaf and Y. C. Yortsos, *Phys. Rev. E* **55**, 7177 (1997); M. Chen, W. Rossen, and Y. C. Yortsos, *Chem. Eng. Sci.* **60**, 4183 (2005).
- [4] P. M. Duxbury, P. D. Beale, and P. L. Leath, *Phys. Rev. Lett.* **57**, 1052 (1986); P. M. Duxbury, P. L. Leath, and P. D. Beale, *Phys. Rev. B* **36**, 367 (1987); S. S. Manna and B. K. Chakrabarti, *ibid.* **36**, 4078 (1987).
- [5] W. Gerstner and W. M. Kistler, *Spiking Neuron Models* (Cambridge University Press, Cambridge, 2002).
- [6] P. A. Calabresi, *Am. Fam. Physician* **70**, 1935 (2004).
- [7] V. Bhat and W. J. Weiner, *Minerva Med.* **96**, 145 (2005).
- [8] P. Erdős and A. Rényi, *Publ. Math. (Debrecen)* **6**, 290 (1959).
- [9] D. J. Watts and S. H. Strogatz, *Nature (London)* **393**, 440 (1998).
- [10] T. H. Cormen, C. E. Leiserson, and R. L. Rivest, *Introduction to Algorithms* (MIT Press, Cambridge, MA, 1990).
- [11] M. Cieplak, A. Maritan, and J. R. Banavar, *Phys. Rev. Lett.* **72**, 2320 (1994).
- [12] L. A. Braunstein, S. V. Buldyrev, R. Cohen, S. Havlin, and H. E. Stanley, *Phys. Rev. Lett.* **91**, 168701 (2003).
- [13] M. Cieplak, A. Maritan, and J. R. Banavar, *Phys. Rev. Lett.* **76**, 3754 (1996).
- [14] N. Schwartz, A. L. Nazaryev, and S. Havlin, *Phys. Rev. E* **58**, 7642 (1998).
- [15] M. Marsili and Y. C. Zhang, *Phys. Rev. E* **57**, 4814 (1998).
- [16] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1994).
- [17] P. Grassberger, *Physica A* **262**, 251 (1999).
- [18] A. Barabasi, *Nature (London)* **435**, 207 (2005).
- [19] Z. Wu, E. López, S. V. Buldyrev, L. A. Braunstein, S. Havlin and H. E. Stanley, *Phys. Rev. E* **71**, 045101(R) (2005).